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ELECTRIC FORCES IN A DIELECTRIC TWO-LAYER CYLINDER WITH NONCONCENTRIC ARRANGEMENT OF LAYERS

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1. Introduction. It is well known that in nonhomogeneous dielectrics placed in an electric field, under the influence of polarization effects electric forces arise that are distinctly manifested in strongly nonhomogeneous dielectrics exposed to high-intensity fields.

In the present work we study electric forces in a two-layer dielectric cylinder of infinite length. In another way, more preferably, this nonhomogeneous structure can be considered as a dielectric cylinder placed into another dielectric cylinder oriented in a parallel but nonconcentric way to the first one (Fig. 1).

The nonhomogeneous system is placed into an external electric field, which is homogeneous and directed transversely to the cylinders' axes at a large distance from them. Under our proposal, the dielectric permittivity constants of all the materials in the system are arbitrary. If in a particular case we let the dielectric permittivity constant of the inside cylinder increase to infinity, we end up with the system of a metal cylinder with a nonhomogeneous insulation coating. Similar systems are characteristic for many electrophysical units and high-voltage apparatus.

In piecewise-homogeneous dielectrics electric fields arise at the boundary surfaces separating the different materials. The surface forces are distributed so that their effects tend to deform the dielectric bodies by drawing them along the external field direction.

A nonuniform distribution of surface forces leads to the emergence of integral forces, which affect the cylinders and tend to displace one cylinder from another.

As a result of local and integral forces, mechanical stresses arise in dielectric structures; these must be taken into account in the design of parts of different electrophysical units for power-generating and technical purposes.

The problem under study requires two separate problems to be solved. First, it is necessary to calculate the electric field inside the cylinders and in the surrounding space. After that, using the results of the calculation of the field, first the surface and then the integral forces can be calculated.

It is noteworthy that both problems have a full analytical solution that can be represented in a simple form. To obtain these solutions, the efficient methods of the theory of functions of a complex variable are used.

2. The Boundary Problem. For the conditions under consideration, the electric field in the system is two-dimensional; therefore, we can turn to the complex variable plane $z = x + iy$.

The electrostatic equations for linear piecewise-homogeneous media allow us to introduce the complex functions of intensity and displacement of electric field

$$E(z) = E_x - iE_y, \quad D(z) = D_x - iD_y, \quad D = \varepsilon E \tag{2.1}$$

(where ε is the dielectric permittivity constant). The functions (2.1) are analytical (excluding the boundary lines) in each of the three regions S_μ ($\mu = 1, 2, 3$) of homogeneous materials (see Fig. 2a).

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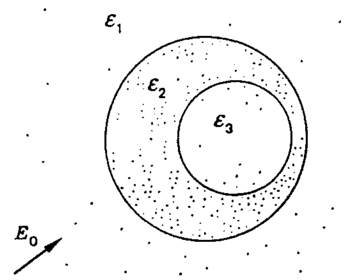

Fig. 1

On the boundaries L_1 and L_2, the conditions of continuity of the tangential components of the electric field intensity vector **E** and the normal components of the displacement vector **D** are satisfied. Written in terms of the function $E(z)$, these conditions have the form

$$\operatorname{Re}\{n(t)\varepsilon_2 E_2(t)\} = \operatorname{Re}\{n(t)\varepsilon_\nu E_\nu(t)\}, \quad \operatorname{Im}\{n(t)E_2(t)\} = \operatorname{Im}\{n(t)E_\nu(t)\}, \quad t \in L_{(\nu+1)/2} \quad (\nu = 1.3), \tag{2.2}$$

where $n(t)$ is a unit normal to the boundaries:

$$n(t) = t/r_1 = e^{i\vartheta} \quad (t \in L_1), \quad n(t) = (t-h)/r_2 \quad (t \in L_2) \quad (0 \leqslant \vartheta < 2\pi)$$

[r_1 and r_2 are the radii of the cylinders ($r_1 > r_2$), h is the distance between the axes of the cylinders]. The indices of the dielectric permittivity constants and the functions of the electric field show that these values belong to the corresponding regions S_μ shown in Fig. 2a.

An additional condition to the problem is the value of the homogeneous external electric field specified in an infinitely distant point

$$E_1(\infty) = E_0 = E_{0x} - iE_{0y}. \tag{2.3}$$

The boundary relations can be somewhat simplified. With this goal, we first write them in an explicit form

$$\begin{aligned}
\varepsilon_2 E_2(t) + \varepsilon_2 \left(\frac{r_1}{t}\right)^2 \overline{E_2(t)} &= \varepsilon_1 E_1(t) + \varepsilon_1 \left(\frac{r_1}{t}\right)^2 \overline{E_1(t)} \quad (t \in L_1),\\
E_2(t) - \left(\frac{r_1}{t}\right)^2 \overline{E_2(t)} &= E_1(t) - \left(\frac{r_1}{t}\right)^2 \overline{E_1(t)} \quad (t \in L_1),\\
\varepsilon_2 E_2(t) + \varepsilon_2 \left(\frac{r_2}{t-h}\right)^2 \overline{E_2(t)} &= \varepsilon_3 E_3(t) + \varepsilon_3 \left(\frac{r_2}{t-h}\right)^2 \overline{E_3(t)} \quad (t \in L_2),\\
E_2(t) - \left(\frac{r_2}{t-h}\right)^2 \overline{E_2(t)} &= E_3(t) - \left(\frac{r_2}{t-h}\right)^2 \overline{E_3(t)} \quad (t \in L_2).
\end{aligned} \tag{2.4}$$

Here the bar over the complex variable denotes complex conjugation (as usual); to write the equalities (2.4), the obvious relations are used:

$$\overline{n(t)} = \frac{\bar{t}}{r_1} = \frac{r_1}{t}, \quad \overline{n(t)} = \frac{\bar{t}-h}{r_2} = \frac{r_2}{t-h}.$$

Excluding the function $\overline{E_2(t)}$ from each couple of equalities (2.4), we obtain two relations

$$\begin{aligned}
2\varepsilon_2 E_2(t) &= (\varepsilon_1 + \varepsilon_2) E_1(t) - (\varepsilon_2 - \varepsilon_1) \left(\frac{r_1}{t}\right)^2 \overline{E_1(t)}, \quad t \in L_1,\\
2\varepsilon_2 E_2(t) &= (\varepsilon_2 + \varepsilon_3) E_3(t) - (\varepsilon_2 - \varepsilon_3) \left(\frac{r_2}{t-h}\right)^2 \overline{E_3(t)}, \quad t \in L_2.
\end{aligned} \tag{2.5}$$

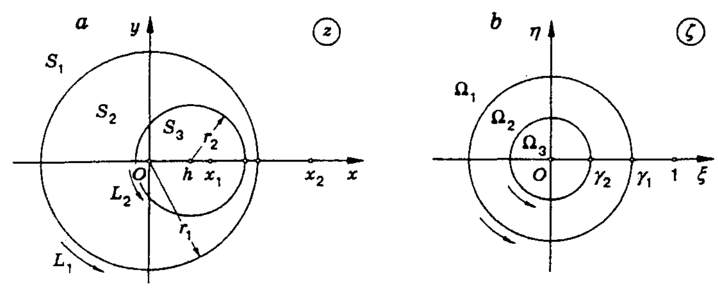

Fig. 2

In the following, it is convenient to use the relative dielectric permittivity constants

$$\Delta_{2\nu} = \frac{\varepsilon_2 - \varepsilon_\nu}{\varepsilon_2 + \varepsilon_\nu}, \quad -1 < \Delta_{2\nu} < 1 \quad (\nu = 1.3). \tag{2.6}$$

Using the parameters (2.6), we can write the boundary relations (2.5) in the following final form:

$$\begin{aligned} (1 + \Delta_{21}) E_2(t) &= E_1(t) - \Delta_{21} \left(\frac{r_1}{t}\right)^2 \overline{E_1(t)}, \qquad t \in L_1, \\ (1 + \Delta_{23}) E_2(t) &= E_3(t) - \Delta_{23} \left(\frac{r_2}{t-h}\right)^2 \overline{E_3(t)}, \qquad t \in L_2. \end{aligned} \tag{2.7}$$

The equalities (2.7) together with the additional condition (2.3) represent the boundary conditions to the generalized homogeneous Riemann boundary problem (recently also called a two-element boundary problem).

The boundary problem (2.3), (2.7) most certainly can be solved. An extensive study of it using the theory of functions of a complex variable is given in the Appendix.

3. Electrostatic Field in the System. The solution to the boundary problem (2.3), (2.7) given in the Appendix allows us to rewrite finally the explicit expressions of the functions of an electric field inside cylinders and in external space.

In the system of coordinates in Fig. 2a, the electric field is determined by the expressions

$$\begin{aligned} E_1(z) &= E_0 + \overline{E_0} \left\{ \frac{\Delta_{21} r_1^2}{z^2} - \frac{1 - \Delta_{21}^2}{\Delta_{21}} (x_2 - x_1)^2 \sum_{k=1}^{\infty} \left[\Delta^k \left(\frac{T_{1k}}{z - Q_{1k}} \right)^2 \right] \right\} \quad (z \in S_1), \\ E_2(z) &= (1 - \Delta_{21}) \left\{ E_0 + (x_2 - x_1)^2 \sum_{k=1}^{\infty} \left\{ \Delta^k \left[E_0 \left(\frac{T_{2k}}{z - Q_{2k}} \right)^2 - \frac{\overline{E_0}}{\Delta_{21}} \left(\frac{T_{1k}}{z - Q_{1k}} \right)^2 \right] \right\} \right\} \quad (z \in S_2), \\ E_3(z) &= (1 - \Delta_{21})(1 + \Delta_{23}) E_0 \left\{ 1 + (x_2 - x_1)^2 \sum_{k=1}^{\infty} \left[\Delta^k \left(\frac{T_{2k}}{z - Q_{2k}} \right)^2 \right] \right\} \quad (z \in S_3). \end{aligned} \tag{3.1}$$

Here

$$\begin{aligned} &\Delta = \Delta_{21}\Delta_{23}, \quad T_{jk} = \frac{a_{jk}}{1 - a_{jk}^2}, \quad Q_{jk} = \frac{x_1 - x_2 a_{jk}^2}{1 - a_{jk}^2} \quad (j = 1.2), \quad a_{1k} = \left[\left(\frac{x_2}{x_1}\right)^{k-1} \left(\frac{x_1 - h}{x_2 - h}\right)^k \right]^{1/2}, \\ &a_{2k} = \frac{1}{a_{1k}} \left(\frac{x_1}{x_2}\right)^{1/2}, \quad x_{1,2} = \frac{1}{2h} \left\{ h^2 + r_1^2 - r_2^2 \mp \left[(h^2 + r_1^2 - r_2^2)^2 - 4 (r_1 h)^2 \right]^{1/2} \right\}. \end{aligned} \tag{3.2}$$

The points x_1 and x_2 are situated in the x axis and are the centers of symmetry with respect to the circles L_1 and L_2.

To illustrate this, in Fig. 3 the patterns of the electric field for the system with the parameters $\varepsilon_1 = 1$, $\varepsilon_2 = 3$, $\varepsilon_3 = 12$, $r_2/r_1 = 0.5$, and $h/r_1 = 0.4$ at three directions of the external electric field ($\alpha = 0$, $\pi/4$, $\pi/2$, and α is the angle between the vector $\mathbf{E}_0$ and the x-axis) are plotted according to formulas (3.1) and (3.2).

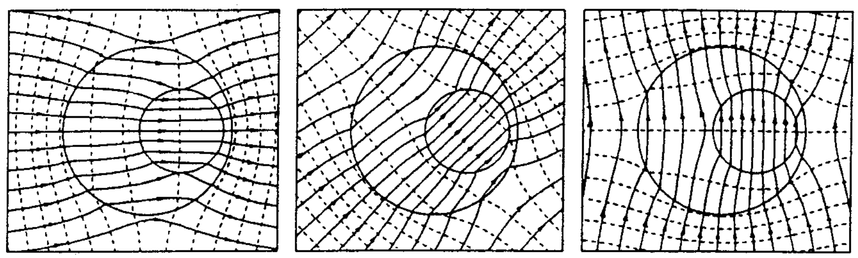

Fig. 3

The expressions for the electric field (3.1) contain a series of linear dipoles, the standard form of which in complex variables is

$$E(z) = \frac{1}{2\pi\varepsilon}\frac{p}{(z-w)^2},$$

where p is the complex moment of the dipole and w is the coordinate of the dipole in the plane of complex variable z.

In expressions (3.1), two groups of dipoles occur. One is situated in the interval $[h,\, x_1)$ of the real axis x with the coordinates Q_{1k}. The density of distribution of the dipoles increases approaching the limit point x_1, which is the point of condensation of the dipoles of this group. The other, with the coordinates Q_{2k}, is also situated on the real axis x with the condensation point at x_2. The dipoles of this group are connected with those of the first group by inverse relations with respect to the circle L_1 (with radius r_1):

$$Q_{1k}Q_{2k} = r_1^2. \tag{3.3}$$

In addition, there is one isolated dipole situated in the center of the circle L_1; it is present only in the expression for $E_1(z)$ and its moment is independent of the radius and the dielectric permittivity constant of the inside cylinder.

Analysis of expressions (3.1) and (3.2) shows that the absolute values of the dipoles from both groups decrease rapidly with the growth of the ordinal number k. This property of the series allows us to limit it to a small number of terms for practical calculations. For example, taking account of first terms only gives us

$$\begin{aligned} E_1(z) &= E_0 + \overline{E_0}\left\{\frac{\Delta_{21}r_1^2}{z^2} - \Delta_{23}(1-\Delta_{21}^2)\frac{r_2^2}{(z-h)^2}\right\}, \quad z \in S_1, \\ E_2(z) &= (1-\Delta_{21})\left\{E_0 + \frac{\Delta(r_1r_2)^2E_0}{h^2(z-r_1^2/h)^2} - \frac{\Delta_{23}r_2^2\overline{E_0}}{(z-h)^2}\right\}, \quad z \in S_2, \\ E_3(z) &= (1-\Delta_{21})(1+\Delta_{23})E_0\left\{1 + \frac{\Delta(r_1r_2)^2}{h^2(z-r_1^2/h)^2}\right\}, \quad z \in S_3. \end{aligned} \tag{3.4}$$

In this approximation, only three dipoles are taken into account: one in the center of the cylinder with large radius, another in the center of the inside cylinder, and the third one situated in the external domain and connected with the second one by inverse relations with respect to the circle with radius r_1.

If in expressions (3.4) or (3.1), (3.2) we set $\Delta_{23} = 0 \quad (\varepsilon_2 = \varepsilon_3)$ or $\Delta_{21} = 0 \quad (\varepsilon_1 = \varepsilon_2)$, we get the well-known solution to the problem of an isolated dielectric cylinder in an external homogeneous electric field.

4. The Surface Forces. In the system under study, free charges do not occur; it is also considered that electrostriction effects do not appear here. In the present case, the appearance of the forces is due to the polarization phenomena in nonhomogeneous dielectrics. The density of these forces is defined by the formula [1, 2]

$$\mathbf{f} = -\frac{1}{2}E^2\,\mathrm{grad}\,\varepsilon. \tag{4.1}$$

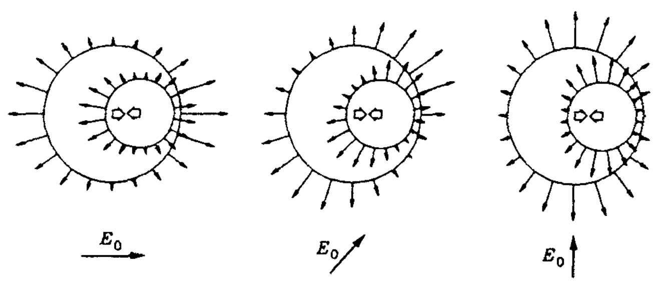

Fig. 4

In piecewise homogeneous media, **f** appears only on boundary surfaces separating different dielectrics. The formula for calculating the density of forces (4.1) in that case is transformed to the form [1]

$$\mathbf{f} = \frac{1}{2}\mathbf{n}\,(\varepsilon_{(-)} - \varepsilon_{(+)})\left[E_{(+)t}^2 + \frac{\varepsilon_{(+)}}{\varepsilon_{(-)}}\,E_{(+)n}^2\right] = \frac{1}{2}\mathbf{n}\,(\varepsilon_{(-)} - \varepsilon_{(+)})\left[E_{(-)t}^2 + \frac{\varepsilon_{(-)}}{\varepsilon_{(+)}}\,E_{(-)n}^2\right]. \tag{4.2}$$

Here **n** is the normal unit vector directed to the domain of the medium denoted by the index $(-)$; E_t and E_n are the tangential and normal components of the vector **E** on the boundary surfaces approaching these from either side, denoted by the indices $(+)$ and $(-)$.

To find the distribution of the density of surface forces on the contours of the cylinders, it is necessary to turn to the expressions obtained for the electric field (3.1), (3.2) and to calculate the normal and tangential components of the vector **E** on the contours L_1 and L_2:

$$E_n = \operatorname{Re}(En), \quad E_t = \operatorname{Im}(En), \quad n = e^{i\vartheta}. \tag{4.3}$$

Substitution of formulas (4.3) into expression (4.2) gives us the desired value of the forces.

As an example, the forces in the system with the parameters $\varepsilon_\mu/\varepsilon_0 = 1,\ 3,\ 12$, $r_2/r_1 = 0.5$, $h/r_1 = 0.4$ at three directions of external electric field ($\alpha = 0,\ \pi/4,\ \pi/2$) were calculated. The results of the calculations illustrating the pattern of distribution of surface force density in the considered system are represented in graphic form in Fig. 4 (in relative values $\mathbf{f}/\varepsilon_0|E_0|^2$).

The surface forces by their action (pressure and extension) tend to deform the cylinders by drawing them along the external electric field.

5. Integral Forces. Nonhomogeneous and nonsymmetric distribution of the surface forces on the cylinders leads to the emergence of integral forces in the system, which tend by their action to displace the cylinders from each other. Forces which are equal in value and directed in an opposite way are applied to the cylinders, which can be calculated (for a unit of length of the cylinders) by taking the integrals of the surface forces on the boundary contours:

$$\mathbf{F}_1 = \int_{L_1} \mathbf{f}\,dt, \quad \mathbf{F}_2 = \int_{L_2} \mathbf{f}\,dt \quad (\mathbf{F}_1 = -\mathbf{F}_2). \tag{5.1}$$

It is necessary to note that the surface forces **f**, according to the formulas (4.1) and (4.2), are directed normal to the boundary circles of the cylinders. Therefore, the resultants of these forces brought to the centers of the circles are exactly the integral forces $\mathbf{F}_1$ and $\mathbf{F}_2$. Hence, the conclusion follows that the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are applied to the centers of the circles (or in space, to the axes of the cylinders).

Calculation of integral forces according to the formulas (5.1) usually requires numerical calculations. In what follows, another approach is used to determine the integral forces in the system, based on the concept of dipole-dipole interactions. Such a method of calculation of the forces allows one to obtain analytical expressions for the integral forces in a simple form convenient for practical analysis.

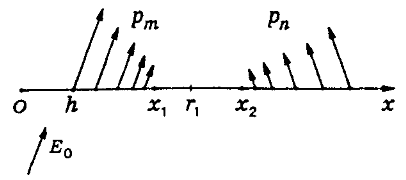

Fig. 5

The initial formula for calculation of the forces in the developed method is the expression for electric field $E_2(z)$ given in (3.1). It is represented by the fields of dipoles of two groups, the properties of which were discussed in Section 3. Schematically, the disposition of the dipoles is shown in Fig. 5 (the orientation of the dipoles corresponds to the case when Δ_{21}, $\Delta_{23} > 0$).

We assume that the medium is homogeneous with dielectric permittivity constant ε_2. In this medium, the interaction of the dipoles is considered, one group of which, with moments p_m ($m = 1, 2, \ldots$), is situated in the interval $[h,\, x_1)$. The coordinates of the dipoles of the second group, with moments p_n ($n = 1, 2, \ldots$), are connected with the coordinates of the dipoles of the first group by inverse transformation with respect to the circle with radius r_1 (3.3).

The forces exerted by the dipoles p_n on the dipoles p_m are defined by the formula

$$\mathbf{F}_{mn} = (p_m \,\mathrm{grad})\, E_n. \tag{5.2}$$

The reciprocal action of the dipoles p_m on the dipoles p_n gives forces equal in value and opposite in direction:

$$\mathbf{F}_{nm} = -\mathbf{F}_{mn}. \tag{5.3}$$

In homogeneous dielectrics formula (5.2) can be transformed to the form

$$\mathbf{F}_{mn} = \mathrm{grad}\,(p_m E_n). \tag{5.4}$$

The summation of the forces exerted by each dipole of one group on all the dipoles of the other group gives the integral forces

$$\mathbf{F}_1 = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \mathbf{F}_{mn}, \qquad \mathbf{F}_2 = -\mathbf{F}_1. \tag{5.5}$$

The general term of the double series (5.5) calculated according to formulas (5.2), (5.3), or (5.4), where p_m and E_n are taken from the expression for $E_2(z)$ in (3.1), has the form

$$\mathbf{F}_{mn} = 4\pi\varepsilon_2 |E_0|^2 \,\frac{\Delta^{m+n}[(1-\Delta_{21})\,(x_2 - x_1)^2 T_{1n} T_{2m}]^2}{\Delta_{21}\,(Q_{2m} - Q_{1n})^3}. \tag{5.6}$$

The forces $\mathbf{F}_{mn}$ turned out to be directed along the x-axis: $\mathbf{F}_{mn} = \mathbf{e}_x F_{xmn}$ ($\mathbf{e}_x$ is the unit vector). The fact that the component F_{ymn} equals zero means that at any direction of the external electric field E_0 the moments in the system are lacking, otherwise, they would tend to turn the cylinders with respect to E_0.

The integral forces, therefore, act along the straight line connecting the centers of the circles. They are the central forces that tend to displace the cylinders from each other, in the given case, along the axis x. The signs of the integral forces indicating the direction of action depend on the relation between the dielectric permittivity constants of the materials in the system.

The double series in formula (5.5) converges rapidly. An approximation taking into account only the few first dipoles from both groups gives good results.

If only the first dipole from each group is taken into account, expressions (5.5) and (5.6), taking account of formulas (3.2), have the form

$$F_1 = F_{11} = -2\varepsilon\Delta_{23}\Delta\,(1-\Delta_{21})^2\,\frac{r^4 h}{(1-h^2)^3} \quad (0 < h < 1 - r). \tag{5.7}$$

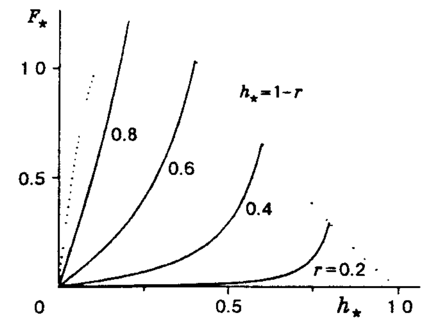

Fig. 6

Expression (5.7) is written in relative values:

$$F_{*11} = F_{11}/F_0 \quad (F_0 = 2\pi r_1 \varepsilon_0 |E_0|^2), \quad \varepsilon_* = \varepsilon_2/\varepsilon_0, \quad r_* = r_2/r_1, \quad h_* = h/r_1$$

(the asterisks are omitted).

Calculation of the integral forces in the system can be done by two methods: 1) direct calculation of the integrals (5.1), which requires a preliminary calculation of the surface forces on the cylinders; 2) on the basis of the dipole-dipole interaction, with simple single-type differentiation operations (5.2) or (5.4) and subsequent computation of the series (5.5).

Both methods of calculation of the integral forces give exactly the same results. However, the second method is easier to perform; it does not require the calculation of the surface forces, and the computation of the electric field in the cylinder of greater radius is sufficient. In addition, the final expressions for the forces are obtained in an analytical form.

The dependence of the integral force on geometrical parameters of the system (r, h) is characterized by the presence of the maximum, which is illustrated in graphical form in Fig. 6, where the values of the relative force at $\varepsilon_1 = 2$, $\varepsilon_2 = 1$, and $\varepsilon_3 = 6$ are plotted. The dotted line enveloping the continuous lines corresponds to the limit case of contact of the cylinders. It has a maximum at $h_* \approx 0.2$ for any values of dielectric permittivities in the system.

6. Conclusion. The study of the integral force in a system of two noncoaxial dielectric cylinders allows one to make the following conclusions.

(1) Integral forces directed from opposite directionc and applied to the cylinders act in the system and tend to displace the cylinders from each other along the straight line connecting the centers of the circles.

(2) The direction of action of the integral forces is determined by the sign of the parameter Δ_{21}, i.e., by the relation between the dielectric permittivities ε_1 and ε_2, and does not depend on the dielectric permittivity of the inside cylinder ε_3.

If $\varepsilon_1 > \varepsilon_2$, the inside cylinder tends to be displaced from the axis of the cylinder of greater radius; if $\varepsilon_1 < \varepsilon_2$, the inside cylinder tends to be displaced to the central position. When the axes of both cylinders coincide, the integral forces equal zero.

(3) The integral force has maximum at $h/r_1 \approx 0.2$.

(4) In any direction of the external electric field the moment in the system is lacking, making it possible for the cylinders to rotate relative to each other.

Appendix. Consider the solution to the boundary problem (2.3), (2.7). As a result, we determine the electric fields inside the cylinders and in external space. The solution to the problem is based on the theory of functions of a complex variable; here the method of conformal mapping and the principle of analytical extension are used.

First, one should map the plane z with nonconcentric circles onto the plane ζ with concentric circles (see Fig. 1). Such a mapping is performed by means of the piecewise-linear function

$$\zeta = T^{-1}(z) = \frac{z - x_1}{z - x_2}, \quad \zeta = \xi + i\eta. \tag{A.1}$$

Here x_1 and x_2 are points symmetric with respect to the circles L_1 and L_2:

$$x_1 x_2 = r_1^2, \quad (x_1 - h)(x_2 - h) = r_2^2.$$

The points x_1 and x_2 are situated in the axis x; their coordinates are given in (3.2).

At mapping (A.1) the regions S_μ turn to the regions Ω_μ ($\mu = 1, 2, 3$), the circles L_1 and L_2 turn to the circles λ_1 and λ_2, and the infinitely distant point $z = \infty$ turns to the point $\xi = 1$ in the real axis ξ.

The function

$$z = T(\zeta) = \frac{x_2\zeta - x_1}{\zeta - 1} \tag{A.2}$$

performs the inverse mapping to (A.1).

In the mapped plane we introduce a piecewise-analytical function $f(\zeta) = E(T(\zeta))$, for the determination of which, according to (2.7) and (A.2), we have the boundary relations

$$\begin{aligned} (1 + \Delta_{21}) f_2(\tau) &= f_1(\tau) - \Delta_{21} \left(\frac{r_1}{T(\tau)}\right)^2 \overline{f_1(\tau)}, \quad \tau \in \lambda_1, \\ (1 + \Delta_{23}) f_2(\tau) &= f_3(\tau) - \Delta_{23} \left(\frac{r_2}{T(\tau) - h}\right)^2 \overline{f_3(\tau)}, \quad \tau \in \lambda_2. \end{aligned} \tag{A.3}$$

From (2.3) follows the additional condition

$$f_1(1) = E_0. \tag{A.4}$$

According to the Laurent theorem, we represent the function $f_2(\zeta)$ analytical in the ring Ω_2 in the form

$$f_2(\zeta) = f_2^+(\zeta) + f_2^-(\zeta), \quad \begin{matrix} f_2^+(\zeta), & |\zeta| < \gamma_1, \\ f_2^-(\zeta), & |\zeta| > \gamma_2, \end{matrix} \tag{A.5}$$

moreover, we can assume without loss of generality that

$$f_2^-(1) = 0. \tag{A.6}$$

Taking representation (A.5) into account, relations (A.3) can be transformed to

$$\begin{aligned} (1 + \Delta_{21}) f_2^+(\tau) + \Delta_{21} \left(\frac{r_1}{T(\tau)}\right)^2 \overline{f_1(\tau)} &= f_1(\tau) - (1 + \Delta_{21}) f_2^-(\tau), \quad \tau \in \lambda_1, \\ (1 + \Delta_{23}) f_2^+(\tau) - f_3(\tau) &= -(1 + \Delta_{23}) f_2^-(\tau) - \Delta_{23} \left(\frac{r_2}{T(\tau) - h}\right)^2 \overline{f_3(\tau)}, \quad \tau \in \lambda_2. \end{aligned} \tag{A.7}$$

In accordance with the principle of continuous analytical continuation, relations (A.4), (A.6), and (A.7) allow one to introduce two analytical functions:

$$\begin{aligned} \Phi(\zeta) &= \begin{cases} (1 + \Delta_{21}) f_2^+(\zeta) + \Delta_{21} \left(\frac{r_1}{T(\zeta)}\right)^2 \overline{f_1\left(\frac{\gamma_1^2}{\bar\zeta}\right)}, & |\zeta| < \gamma_1, \\ f_1(\zeta) - (1 + \Delta_{21}) f_2^-(\zeta), & |\zeta| > \gamma_1, \end{cases} \\ \Psi(\zeta) &= \begin{cases} -(1 + \Delta_{23}) f_2^+(\zeta) + f_3(\zeta), & |\zeta| < \gamma_2, \\ (1 + \Delta_{23}) f_2^-(\zeta) + \Delta_{23} \left(\frac{r_2}{T(\tau) - h}\right)^2 \overline{f_3\left(\frac{\gamma_2^2}{\bar\zeta}\right)}, & |\zeta| > \gamma_2. \end{cases} \end{aligned} \tag{A.8}$$

Here, with regard to (A.2),

$$\frac{r_1}{T(\zeta)} = \gamma_1 \frac{\zeta - 1}{\zeta - \gamma_1^2}, \quad \frac{r_2}{T(\zeta) - h} = \gamma_2 \frac{\zeta - 1}{\zeta - \gamma_2^2}. \tag{A.9}$$

According to expressions (A.8) and (A.9), the function $\Phi(\zeta)$ has a pole of no higher than second order in the point $\zeta = \gamma_1^2$, and is, therefore, a polynomial of second order, which is convenient to represent in the form

$$\Delta_{21}\left[C_2\left(\frac{r_1}{T(\zeta)}\right)^2 + C_1\frac{r_1}{T(\zeta)} + C_0\right], \tag{A.10}$$

where C_0, C_1, and C_2 are complex constants to be determined.

Formulas (A.6), (A.8), and (A.9) imply that in the point $\zeta = 1$ the function $\Psi(\zeta)$ equals zero, and, therefore, according to the Liouville theorem,

$$\Phi(\zeta) \equiv 0.$$

Combining formulas (A.8), (A.10), and (A.11), we obtain four equalities:

$$\begin{gathered}(1+\Delta_{21})\, f_2^+(\zeta) + \Delta_{21}\left(\frac{r_1}{T(\zeta)}\right)^2 \overline{f_1\left(\frac{\gamma_1^2}{\bar\zeta}\right)} = \Delta_{21}\left[C_2\left(\frac{r_1}{T(\zeta)}\right)^2 + C_1\frac{r_1}{T(\zeta)} + C_0\right], \quad |\zeta| < \gamma_1,\\ f_1(\zeta) - (1+\Delta_{21})\, f_2^-(\zeta) = \Delta_{21}\left[C_2\left(\frac{r_1}{T(\zeta)}\right)^2 + C_1\frac{r_1}{T(\zeta)} + C_0\right], \quad |\zeta| > \gamma_1,\\ (1+\Delta_{23})\, f_2^+(\zeta) - f_3(\zeta) = 0, \quad |\zeta|<\gamma_2, \qquad (1+\Delta_{23})\, f_2^-(\zeta) + \Delta_{23}\left(\frac{r_2}{T(\tau)-h}\right)^2 \overline{f_3\left(\frac{\gamma_2^2}{\bar\zeta}\right)} = 0, \quad |\zeta| > \gamma_2.\end{gathered} \tag{A.12}$$

Then we can determine the constants C_0, C_1, and C_2.

Assuming that in the second equality from (A.12) $\zeta = 1$ and with regard to formulas (A.4), (A.6), and (A.9), we obtain

$$C_0 = E_0/\Delta_{21}. \tag{A.13}$$

The first equality from (A.12) by inverse transformation with respect to the circle γ_1 can be brought to the form

$$(1+\Delta_{21})\, \overline{f_2^+\left(\frac{\gamma_1^2}{\bar\zeta}\right)} + \Delta_{21}\left(\frac{T(\zeta)}{r_1}\right)^2 f_1(\zeta) = \Delta_{21}\left[\overline{C}_2\left(\frac{T(\zeta)}{r_1}\right)^2 + \overline{C}_1\frac{T(\zeta)}{r_1} + \overline{C}_0\right], \quad |\zeta| > \gamma_1.$$

Hence,

$$(1+\Delta_{21})\left(\frac{r_1}{T(\zeta)}\right)^2 \overline{f_2^+\left(\frac{\gamma_1^2}{\bar\zeta}\right)} + \Delta_{21} f_1(\zeta) = \Delta_{21}\left[\overline{C}_2 + \overline{C}_1\frac{r_1}{T(\zeta)} + \overline{C}_0\left(\frac{T(\zeta)}{r_1}\right)^2\right], \quad |\zeta| > \gamma_1.$$

Taking the formulas (A.4) and (A.9) into account and substituting the value $\zeta = 1$, we have

$$C_2 = \overline{E}_0. \tag{A.14}$$

One can also prove (we omit the computations for brevity) that

$$C_1 = 0. \tag{A.15}$$

Thus, the constants in the first and the second equalities from (A.12) are determined by the formulas (A.13)–(A.15), and the equalities can be finally written as

$$\begin{gathered}(1+\Delta_{21})\, f_2^+(\zeta) + \Delta_{21}\left(\frac{r_1}{T(\zeta)}\right)^2 \overline{f_1\left(\frac{\gamma_1^2}{\bar\zeta}\right)} = \Delta_{21}\overline{E}_0\left(\frac{r_1}{T(\zeta)}\right)^2 + E_0, \; |\zeta| < \gamma_1,\\ f_1(\zeta) - (1+\Delta_{21})\, f_2^-(\zeta) = \Delta_{21}\overline{E}_0\left(\frac{r_1}{T(\zeta)}\right)^2 + E_0, \quad |\zeta| > \gamma_1.\end{gathered} \tag{A.16}$$

On the basis of the second equality from (A.16), expression (A.5), and the third equality from (A.12), we can represent the desired function $f(\zeta)$ as

$$f(\zeta)=\begin{cases} f_1(\zeta)=(1+\Delta_{21})\,f_2^-(\zeta)+\Delta_{21}\overline{E}_0\left(\gamma_1\frac{\zeta-1}{\zeta-\gamma}\right)^2+E_0, & |\zeta|>\gamma_1,\\ f_2(\zeta)=f_1^+(\zeta)+f_2^-(\zeta), & \gamma_2<|\zeta|<\gamma_1,\\ f_3(\zeta)=(1+\Delta_{23})\,f_2^+(\zeta), & |\zeta|<\gamma_2.\end{cases} \tag{A.17}$$

The following calculations are connected with the finding of the function $f_2^-(\zeta)$, and then $f_2^+(\zeta)$, which will determine the function $f(\zeta)$.

We perform inverse transformation in the second equality from (A.16), and then combine it with the first one. As a result, we obtain

$$f_2^+(\zeta)+\Delta_{21}\left(\frac{r_1}{T(\zeta)}\right)^2\overline{f_2^-\left(\frac{\gamma_1^2}{\bar{\zeta}}\right)}=(1-\Delta_{21})\,E_0,\quad |\zeta|<\gamma_1. \tag{A.18}$$

Similarly, combination of the last two equalities from (A.12) gives

$$\overline{f_2^-\left(\frac{\gamma_2^2}{\bar{\zeta}}\right)}+\Delta_{23}\left(\frac{T(\zeta)-h}{r_2}\right)^2 f_2^+(\zeta)=0,\quad |\zeta|<\gamma_2. \tag{A.19}$$

We can exclude the function $f_2^+(\zeta)$ from the relations (A.18) and (A.19) and make use of the formulas (A.9). Then,

$$\Delta\left(\gamma_1\frac{\zeta-1}{\zeta-\gamma_1^2}\right)^2\overline{f_2^-\left(\frac{\gamma_1^2}{\bar{\zeta}}\right)}-\left(\gamma_2\frac{\zeta-1}{\zeta-\gamma_2^2}\right)^2\overline{f_2^-\left(\frac{\gamma_2^2}{\bar{\zeta}}\right)}=\Delta_{23}\,(1-\Delta_{21})\,E_0,\ |\zeta|<\gamma_2. \tag{A.20}$$

Performing the inverse transformation of the equality (A.20), we obtain a functional equation with respect to $f_2^-(\zeta)$:

$$f_2^-(\zeta)=-\Delta_{23}\,(1-\Delta_{21})\,\overline{E}_0\left(\gamma_2\frac{\zeta-1}{\zeta-\gamma_2^2}\right)^2+\Delta\left(\Gamma^{-1}\frac{\zeta-1}{\zeta-\Gamma^{-2}}\right)^2 f_2^-(\Gamma^2\zeta),\ |\zeta|>\gamma_2. \tag{A.21}$$

Here $\Gamma=\gamma_1/\gamma_2$.

The solution of Eq. (A.21) can be found by the method of mathematical induction with use of successive substitutions of the relations

$$f_2^-(\Gamma^2\zeta)=-\Delta_{23}\,(1-\Delta_{21})\,\overline{E}_0\left(\gamma_2\frac{\zeta-\Gamma^{-2}}{\zeta-\gamma_2^2\Gamma^{-2}}\right)^2+\Delta\left(\Gamma^{-1}\frac{\zeta-\Gamma^{-2}}{\zeta-\Gamma^{-4}}\right)^2 f_2^-(\Gamma^4\zeta),\quad |\zeta|>\gamma_2,$$

$$f_2^-(\Gamma^4\zeta)=-\Delta_{23}\,(1-\Delta_{21})\,\overline{E}_0\left(\gamma_2\frac{\zeta-\Gamma^{-4}}{\zeta-\gamma_2^2\Gamma^{-4}}\right)^2+\Delta\left(\Gamma^{-1}\frac{\zeta-\Gamma^{-4}}{\zeta-\Gamma^{-6}}\right)^2 f_2^-(\Gamma^6\zeta),\quad |\zeta|>\gamma_2$$

etc.

After n applications of this procedure, we obtain

$$f_2^-(\zeta)=-\frac{1-\Delta_{21}}{\Delta_{21}}\,\overline{E}_0\,(\zeta-1)^2\sum_{k=1}^{n}\left[\Delta^k\left(\frac{\gamma_1\Gamma^{-k}}{\zeta-\gamma_1^2\Gamma^{-2k}}\right)^2\right]+\Delta^n\left(\Gamma^{-n}\frac{\zeta-1}{\zeta-\Gamma^{-2n}}\right)^2 f_2^-(\Gamma^{2n}\zeta),\ |\zeta|>\gamma_2. \tag{A.22}$$

. The remainder term of the series (A.22) contains small parameters Δ, $\Gamma^{-1}<1$ and, therefore, at unlimited increase of the number n, goes to zero.

Thus, the function $f_2^-(\zeta)$ has the form

$$f_2^-(\zeta)=-\frac{1-\Delta_{21}}{\Delta_{21}}\,\overline{E}_0\,(\zeta-1)^2\sum_{k=1}^{\infty}\left[\Delta^k\left(\frac{\gamma_1\Gamma^{-k}}{\zeta-\gamma_1^2\Gamma^{-2k}}\right)^2\right],\quad |\zeta|>\gamma_2. \tag{A.23}$$

The function $f_2^+(\zeta)$ is determined by use of (A.18) and (A.23) (after the inverse transformation with respect to the circle γ_1). The calculations give

$$f_2^+(\zeta) = (1 - \Delta_{21}) E_0 \left\{ 1 + (\zeta - 1)^2 \sum_{k=1}^{\infty} \left[\Delta^k \left(\frac{\Gamma^k}{\zeta - \Gamma^{2k}} \right)^2 \right] \right\}, \quad |\zeta| < \gamma_1. \tag{A.24}$$

Substitution of expressions (A.23) and (A.24) for relations (A.17) determines the desired function $f(\zeta)$. By following this with a transition to the plane z, we obtain expressions for the electric field given in Section 3.

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